THE EPQ WITH PARTIAL BACKORDERING: A NEW APPROACH

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ABSTRACT

Previous authors have shown that if demand that cannot be filled from stock is partially backordered, then using the full-backordering model or assuming that all stockouts will result in lost sales can substantially increase the cost relative to using a model that recognizes the percentage of the stockouts that will be backordered. In this paper we extend our previous work on the deterministic EOQ with partial backordering to develop a model for the EPQ with partial backordering that results in equations more like those for the full-backordering extension of the basic EPQ with full backordering than those for a previous model.

Keywords: Inventory Theory, EPQ, Partial Backordering

INTRODUCTION

Any inventory control system has to determine when and how much to order. The model that is by far the best known is the classic square-root economic order quantity (EOQ) model. While the reasonableness of this model’s assumptions has been criticized, it is widely and successfully used in practice. Further, it forms the basis for many other models that relax or adapt one or more of its assumptions, including the continuous receipt or EPQ model.

A key assumption of the basic EOQ and EPQ models is that stockouts are not permitted. However, if customers are always willing to wait for delivery, planned backorders may make economic sense, even if they incur a cost. Relaxing the basic models’ assumption that stockouts are not permitted led to the development of models for the two basic cases for stockouts: backorders and lost sales. What took longer to develop was a model for the case where only a percentage of customers are willing to wait for delivery and the rest will cancel their orders unless the supplier fills them by using more expensive alternative supply methods.

After reviewing models for the cases of all backorders and all lost sales, we briefly summarize five models for the EOQ model with partial backordering and the only paper we are aware of for the EPQ model with partial backordering. Then we present an alternative approach to modeling the problem and determining equations for when and how much to order.

NOTATION AND TERMINOLOGY
**Parameters:**

- \( D \) = demand per year  
- \( P \) = production rate per year if continuously producing  
- \( s \) = the unit selling price  
- \( C_o \) = the fixed cost of placing and receiving an order  
- \( C_p \) = the variable cost of a purchasing or producing a unit  
- \( C_h \) = the cost to hold a unit in inventory for a year  
- \( C_b \) = the cost to keep a unit backordered for a year  
- \( C_g \) = the goodwill loss on a unit of unfilled demand  
- \( C_l = (s - C_p) + C_g \) = the cost for a lost sale, including the lost profit on that unit and any goodwill loss  
- \( \beta \) = the fraction of stockouts that will be backordered

**Variables:**

- \( Q \) = the order quantity  
- \( T \) = the time between orders or the length of an order cycle  
- \( I \) = the maximum inventory  
- \( S \) = the maximum stockout level, including both backorders and lost sales  
- \( B \) = the maximum backorder position (\( B = \beta S \) )  
- \( F \) = the fill rate or the percentage of demand that will be filled from stock

**THE “PURE” STOCKOUT CASES: ALL BACKORDERS AND ALL LOST SALES**

**The All Backorder Case**

The model under the assumption that all stockouts are backordered at a cost \( C_b \) per unit per year for the time they are backordered appears in many basic texts. The optimal values for \( Q^* \), the order quantity, \( B^* \), the maximum backorder quantity, and \( T^* \), the time between orders, are:

\[
Q^* = \sqrt{\frac{2C_o D}{C_h (1-D/P)}} \frac{C_h + C_b}{C_b}, \quad B^* = Q^* \left(1 - \frac{D}{P}\right) \left(\frac{C_h}{C_h + C_h}\right), \quad T^* = \sqrt{\frac{2C_o}{DC_h (1-D/P)}} \frac{C_b + C_h}{C_b} \quad (1)
\]

Thus \( Q^* \) and \( T^* \) are those given by the basic EPQ formula inflated by a term that reflects the relative sizes of the unit inventory cost per year and the unit backorder cost per year and \( B^* \) is a fraction of \( Q^* \) that depends on the relative sizes of those two costs and the relative production and demand rates.

**The All Lost Sales Case**

Zipkin [7] shows for the basic EOQ that if demands occurring during a stockout period are lost sales rather than backorders, the optimal policy is to either have no stockouts or all stockouts, depending on which costs less. The same result can be proved for the EPQ.

**MODELS FOR THE PARTIAL BACKORDERING EOQ**

Since it is optimal to allow some stockouts if all customers will wait (\( \beta = 1 \)) and it is optimal to either allow no stockouts or lose all sales if no customers will wait (\( \beta = 0 \)), it is logical that there
will be a value of $\beta$ below which one should use the optimal ordering policy for the lost-sales case—either using the basic EOQ model or never ordering at all, depending on which alternative is less costly—and above which one should allow stockouts, some of which will be backordered. Determining an optimal policy for the partial backordering EOQ problem starts with determining the minimum value of $\beta$ for which stockouts should be allowed and, if $\beta$ is greater than this minimum value, determining the optimal order quantity.

Models for the partial backordering EOQ problem were developed by Montgomery et al. [2], Rosenberg [5], Park [3], San José et al. [6], and Pentico and Drake [4]. These papers took somewhat different approaches to modeling the problem, differing primarily in which decision variables they focused on, although San José et al. [6], rather than assuming that $\beta$ is a constant, considered a number of different “customer impatience” functions that have the property that the percentage backordered does not decrease as the replenishment date approaches. Each of the first four papers, however, resulted in equations or procedures that are somewhat difficult to use for computing the relevant decision variable values. Pentico and Drake [4] used a different set of variables that resulted in equations that are more comparable to those of the model for the EOQ with full backordering.

**Modeling Inventory For The EPQ With Backordering**

The only model we have found for the EPQ with partial backordering is by Mak [1]. Since his approach to the treatment of demands that occur while there is no stock but the production run has started differs from the approach we will use, we first discuss that issue.

With partial backordering, from the time the system runs out of stock until the time the next order is received (EOQ) or the next production run begins (EPQ), a fraction $\beta$ of incoming demand will be backordered until the maximum backorder level $B = \beta S$ is reached.

In the EOQ models with full or partial backordering, the entire order quantity $Q$ is received at the same time, so all the backorders can be filled at once, with the maximum inventory rising immediately to $I = Q - B$. In the EPQ model with full backordering, the order quantity $Q$ is received in a constant stream at a rate of $P$. Since demands that occur during the time it takes to fill all the backorders are also backordered if they are not filled immediately, it makes no difference whether the incoming orders are filled before the backorders (LIFO) or the backorders are filled before the incoming orders (FIFO). Inventory increases at the rate of $P - D$ until the backorder is eliminated and the maximum inventory level, $I = Q(1 - P/D) - B$, is reached.

For the EPQ with partial backordering, however, whether LIFO or FIFO is used to determine the order in which demands are filled after the production run begins can make a difference in the net inventory level. Whether it does or not depends on the answer to an additional question: What happens to the demands that occur when there is no stock on hand but the production run has been started? If one assumes that incoming demands will be filled before the existing backorders (LIFO) and further assume that none of the existing backorders will convert to lost sales, then the net inventory level for the EPQ with partial backordering will increase at a rate of $(1 - D/P)$. This is Mak’s [1] unstated assumption. If, however, one assumes, as we do, that the existing backorders will be filled before any new demands (FIFO) and further assume that only a fraction $\beta$ of these new orders that cannot immediately be filled will be backordered, with the
rest being lost sales, then the net inventory level will increase at a rate of \((1 - \beta D/P)\) until the backorder is eliminated and then will increase at a rate of \((1 - D/P)\) until the maximum inventory level is reached. (If all incoming orders will wait once the production run has started, it makes no difference whether LIFO or FIFO is used.)

**Mak’s [1] Model For The EPQ With Partial Backordering**

Mak’s assumptions are the usual ones for the EPQ model with full backordering except that only a fraction \(\beta\) of the stockouts will be backordered, with the rest being lost sales. As noted above, he implicitly assumes that there will be no increase in either backorders or lost sales once the production phase begins, so the backorders are filled at a rate of \(P - D\).

Mak’s decision variables are \(T\), the length of an inventory cycle, and \(t\), the length of time from when the inventory level reaches 0 until the next production run begins. The cost function he develops is convex, and thus the optimal solution can be found by setting the two partial derivatives equal to 0 and solving the resulting equations simultaneously. He does this by developing an equation for \(T\) as a function of \(t\) and then, by using this to eliminate \(T\) from one of the equations, finds an expression for \(t^*\) as a function of the parameters and then, using this, finds an expression for \(T^*\). Both of these equations are quite complicated. It must also be noted that, although Mak develops a statement of a condition that \(\beta\) must satisfy for the partial backordering EPQ equations to apply, that condition is not as simple as the ones developed for the partial backordering EOQ models or as the one to be developed here.

**A DIFFERENT APPROACH**

We use the same assumptions about costs and demand as used in the basic EOQ with full backordering model and as used by Mak [1]. However, we assume that a FIFO policy is used to fill backorders once the production run starts. As in Pentico and Drake [4], we use the variables \(T\), the length of an order cycle, and \(F\), the fraction of demand to be filled from stock. Using these variables, the function for the average profit per year is:

\[
\Gamma(T,F) = \frac{Co}{T} + \frac{C_h D T F^2}{2} (1 - \frac{D}{P}) + \frac{\beta C_b D T (1 - F)^2}{2} (1 - \frac{D}{P}) + C_i D (1 - \beta)(1 - F).
\]  

(2)

Replacing \(C_h (1 - D/P)\) by \(C'_h\) and \(C_h (1 - \beta D/P)\) by \(C'_b\) and taking the partial derivative of \(\Gamma(T,F)\) with respect to \(T\) and setting it equal to 0 gives:

\[
\frac{\partial \Gamma}{\partial T} = -\frac{C_o}{T^2} + \frac{C'_h D F^2}{2} + \frac{\beta C'_b D (1-F)^2}{2} = 0.
\]

This gives, after some algebra:

\[
T^2 = \frac{2C_o}{D[C'_h F^2 + \beta C'_b (1-F)^2]} \quad \text{and} \quad T(F) = \sqrt[2]{\frac{2C_o}{D[C'_h F^2 + \beta C'_b (1-F)^2]}}.
\]  

(3)
Taking the partial derivative of $\Gamma(T,F)$ with respect to $F$ and setting it equal to 0 gives:

$$\frac{\partial \Gamma}{\partial F} = C'_h DTF - \beta C'_i DT(1 - F) - (1 - \beta)C_lD = 0.$$ 

After some algebra, this results in:

$$F(T) = \frac{(1 - \beta)C'_i + \beta C'_b T}{T(C'_h + \beta C'_b)}.$$  \hspace{1cm} (4)

Substituting this expression for $F$ into equation (3), we get, after some algebra:

$$T^* = \frac{2C_o}{DC_h} \left[ \frac{C'_h + \beta C'_b}{\beta C'_b} \right] - \frac{[(1 - \beta)C_i]^2}{\beta C'_h C'_b}.$$  \hspace{1cm} (5)

We note that, with the replacement of $C_h$ by $C'_h$ and $C_b$ by $C'_b$, these are the same equations for $T^*$ and $F(T^*)$ as were found in Pentico and Drake [4] for the EOQ with partial backordering.

$T^*$ for the partial backordering model must be at least as large as $T^*$ for the basic EPQ, so:

$$\frac{2C_o}{DC_h} \left[ \frac{C'_h + \beta C'_b}{\beta C'_b} \right] - \frac{[(1 - \beta)C_i]^2}{\beta C'_h C'_b} \geq \frac{2C_o}{DC_h}.$$ 

After some algebra, this leads us to the following conclusion: For the equations for $T^*$ and $F^*$ to be give the optimal solution, we must have:

$$\beta \geq \beta^* = 1 - \sqrt{\frac{2C_o C'_h}{DC_i^2}}.$$  \hspace{1cm} (6)

We note that, with the replacement of $C_h$ by $C'_h$, this is the same condition derived by Park [3] and Pentico and Drake [4] for the EOQ with partial backordering.

It is interesting, and encouraging, to note that the form of the equation for $T^*$ in (5) is similar to the equation for $T^*$ for the full-backordering case given in (2): Similarly, the equation for the optimal value of $F^*$ in (4) is logical in that it reflects the relative sizes of the cost of not filling a unit of demand from stock and the cost of filling a unit of demand, whether immediately from stock or eventually by being backordered.

**Procedure for determining the optimal values for $T$, $F$, $Q$, $I$, $S$, and $B$**

1. Determine $\beta^*$, the critical value for $\beta$, from (6).
2. a. If $\beta \leq \beta^*$, determine $T$ from the basic EPQ model and determine the optimal cost of allowing no stockouts. Compare this with the cost of losing all demand, $C_lD$, to determine whether to allow no stockouts or all stockouts.
b. If $\beta > \beta^*$, use (5) to determine the value of $T^*$ and substitute it into (4) to determine the value of $F^*$.

3. Determine the values of the other variables as follows:
   - Total demand during a cycle = $DT^*$
   - Maximum inventory level = $I^* = F^*DT^*(1 - D/P)$
   - Maximum stockout level = $S^* = (1 - F^*)DT^*(1 - \beta D/P)$
   - Maximum backorder level = $B^* = \beta S^*$
   - Order quantity = $Q^* = F^*DT^* + B^*$

CONCLUSION

As noted by several previous authors in the context of the EOQ model, determining the optimal ordering and stockout quantities when demands that cannot be filled from stock are partially backordered is a much more complicated problem than for the cases in which all stockouts are either backordered or result in lost sales. As shown here, the same is true for the EPQ model. However, by changing the decision variables from $Q$, the order quantity, and $S$, the stockout level, to $T$, the time between orders, and $F$, the fill rate, we have developed a model with equations that are more like those for the basic EPQ model and its full-backordering extension and are much easier to solve than the equations developed by Mak [1].

REFERENCES


