THE EPQ WITH PARTIAL BACKORDERING AND A LIFO POLICY ON FILLING BACKORDERS

David W. Pentico, School of Business, Duquesne University, Pittsburgh, PA 15282-0180, pentico@duq.edu, 412-396-6252
Matthew J. Drake, School of Business, Duquesne University, Pittsburgh, PA 15282-0180, drake987@duq.edu, 412-396-1959

ABSTRACT

The first published model for the EPQ when only a percentage of stockouts will be backordered was by Mak [1]. He assumed that, once production started, new customer orders would be filled before the existing backorders. The resulting equation set was complicated and non-intuitive. In this paper we use the approach taken by Pentico and Drake [4] for the EOQ with partial backordering to develop equations for the EPQ with partial backordering and a LIFO policy on filling backorders that are more like those for the EPQ with full backordering.

INTRODUCTION

A key assumption of the basic EOQ and EPQ models is that stockouts are not permitted. However, if customers are always willing to wait for delivery, planned backorders can make economic sense, even if they incur a cost. Relaxing the basic models’ assumption that stockouts are not permitted led to the development of models for the two basic stockout cases: backorders and lost sales. What took longer to develop was a model for the case where only a percentage of customers are willing to wait for delivery and the rest will cancel their orders unless the supplier fills them within the normal delivery time by using more expensive supply methods.

After reviewing the models for the “pure” stockout cases of backorders and lost sales, we will briefly discuss the EPQ models with partial backordering by Mak [1] and Pentico and Drake [5]. We will present an alternative to Mak’s approach to modeling the EPQ with a LIFO policy on filling backorders based on the approach in Pentico and Drake [4,5].

NOTATION AND TERMINOLOGY

Parameters

\( D = \) demand per year
\( P = \) production rate per year if constantly producing
\( s = \) the unit selling price
\( C_0 = \) the fixed cost of placing and receiving an order
\( C_p = \) the variable cost of a purchasing or producing a unit
\( C_h = \) the cost to hold a unit in inventory for a year
\( C_b = \) the cost to keep a unit backordered for a year
\( C_g = \) the goodwill loss on a unit of unfilled demand
\( C_l = (s - C_p) + C_g = \) the cost for a lost sale, including the lost profit on that unit and any goodwill loss
\[ \beta = \text{the fraction of stockouts that will be backordered} \]

**Variables**
- \( Q = \text{the order quantity} \)
- \( T = \text{the length of an order cycle} \)
- \( I = \text{the maximum inventory level, with } \bar{I} \text{ being the average inventory level over the year} \)
- \( S = \text{the maximum stockout level, including both backorders and lost sales} \)
- \( B = \text{the maximum backorder position, with } \bar{B} \text{ being the average backorder level over the year (} B = \beta S) \)
- \( F = \text{the fill rate or the percentage of demand that will be filled from stock} \)

**THE “PURE” STOCKOUT CASES: BACKORDERS AND LOST SALES**

The EPQ model under the assumption that all stockouts are backordered at a cost \( C_b \) per unit per year appears in many basic texts. The optimal values for \( Q^* \), the order quantity, \( B^* \), the maximum backorder quantity, and \( T^* \), the time between orders, are:

\[
Q^* = \frac{2C_o D}{\sqrt{C_h(1-D/P)}} \sqrt{\frac{C_h + C_b}{C_b}},
B^* = Q^*(1 - \frac{D}{P}) \left( \frac{C_h}{C_b + C_h} \right),
T^* = \frac{2C_o}{DC_h(1-D/P)} \sqrt{\frac{C_h + C_b}{C_b}}
\]

It is possible to prove for the basic EOQ that if demands occurring during a stockout period are lost sales rather than backorders, the optimal policy is to either have no stockouts or all stockouts, depending on which costs less. The same result can be proved for the EPQ.

**MODELS FOR THE EOQ AND EPQ WITH PARTIAL BACKORDERING**

Since, for the EOQ, it is optimal to allow some stockouts if all customers will wait (\( \beta = 1 \)) and it is optimal to either allow no stockouts or lose all sales if no customers will wait (\( \beta = 0 \)), it is logical that there will be a value of \( \beta \) below which one should use the optimal ordering policy for the lost-sales case – either using the basic EOQ model or never ordering at all, depending on which alternative is less costly – and above which one should allow stockouts, some of which will be backordered. Determining an optimal policy for the partial backordering EOQ problem starts with determining the minimum value of \( \beta \) for which stockouts should be allowed and, if \( \beta \) is greater than this minimum value, determining the optimal order quantity.

Models for the partial backordering EOQ problem were developed by Montgomery et al. [2], Rosenberg [6], Park [3], San José et al. [7], and Pentico and Drake [4]. These papers took somewhat different approaches to modeling the problem, differing primarily in which decision variables they focused on, although San José et al. [7], rather than assuming that \( \beta \) is a constant, considered a number of different “customer impatience” functions that have the property that the percentage backordered does not decrease as the replenishment date approaches. The first four papers resulted in complicated solution procedures. Pentico and Drake [4] took a different approach that resulted in a model that is more comparable to the EOQ with full backordering.
Mak [1] and Pentico and Drake [5] developed models for the EPQ with partial backordering. Other than notation, they differ in their policies on filling demands that occur while there is no stock but the production run has started, which has implications for modeling the problem.

**Modeling Inventory For The EPQ With Backordering**

With partial backordering, from the time the system runs out of stock until the time the next order is received (EOQ) or the next production run begins (EPQ), a fraction $\beta$ of incoming demand will be backordered until the maximum backorder level $B = \beta S$ is reached.

In the EOQ models with full or partial backordering, the entire order quantity $Q$ is received at the same time, so all the backorders can be filled at once, with the maximum inventory rising immediately to $I = Q - B$. In the EPQ model with full backordering, the order quantity $Q$ is received in a constant stream at a rate of $P$. Since all demands that occur during the time it takes to fill all the backorders are also backordered if they are not filled immediately, it makes no difference whether the incoming orders are filled before the backorders (LIFO) or the backorders are filled before the incoming orders (FIFO). Inventory increases at the rate of $P - D$ until the backorder is eliminated and the maximum inventory level, $I = Q(1 - P/D) - B$, is reached.

For the EPQ with partial backordering, however, whether LIFO or FIFO is used to determine the order in which demands are filled after the production run begins can make a difference in the net inventory level. Whether it does or not depends on the answer to an additional question: What happens to the demands that occur when there is no stock on hand but the production run has been started? If one assumes that incoming demands will be filled before the existing backorders (LIFO) and further assume that none of the existing backorders will convert to lost sales, then the net inventory level for the EPQ with partial backordering will increase at a rate of $P - D$. This is Mak’s [1] unstated assumption. If one assumes, as Pentico and Drake [5] did, that the existing backorders will be filled before any new demands (FIFO) and further assume that only a fraction $\beta$ of these new orders that cannot immediately be filled will be backordered, with the rest being lost sales, then the net inventory level will increase at a rate of $P - \beta D$ until the backorder is eliminated and then will increase at a rate of $P - D$ until the maximum inventory level is reached. (If all incoming orders will wait once the production run has started, it makes no difference whether LIFO or FIFO is used.)

**Mak’s [1] model for the EPQ with partial backordering**

Mak’s assumptions are the usual ones for the EPQ model with full backordering except that only a fraction $\beta$ of the stockouts will be backordered, with the rest being lost sales. As noted above, he implicitly assumes that there will be no increase in either backorders or lost sales once the production phase begins, so the backorders are filled at a rate of $P - D$.

Mak’s decision variables are $T$, the length of an inventory cycle, and $t$, the length of time from when the inventory level reaches 0 until the next production run begins (which is $t_1$ in the analysis to follow). His cost function is convex, so the optimal solution can be found by setting the two partial derivatives equal to 0 and solving the resulting equations simultaneously. He develops an equation for $T$ as a function of $t$ and, using this to eliminate $T$ from one of the equations, finds an expression for $t^*$ as a function of the parameters, which he uses to find an
Pentico & Drake’s [5] model for the EPQ with Partial Backordering
In contrast with Mak [1], Pentico and Drake assumed a FIFO policy on filling backorders. Using $T$ and $F$ as their decision variables, they developed the following equations for an optimal policy:

$$T^* = \sqrt{\frac{2C_a}{DC_h} \left[ \frac{C_h + \beta C_b}{\beta C_b} \right] - \frac{[(1-\beta)C_i]^2}{\beta C_h C_b}}$$

and

$$F(T) = \frac{(1-\beta)C_i + \beta C_b T}{T(C_b + \beta C_b)}$$

(2)

where $C^i_h = C_h(1 - D/P)$ and $C^i_b = C_b(1 - \beta D/P)$. They also developed a statement of the condition that $\beta$ must meet for partial backordering to be optimal which, with the replacement of $C^i_h$ by $C_h$, is the same as in Pentico and Drake [4] for the EOQ with partial backordering:

$$\beta \geq \beta^* = 1 - \frac{2C_a C^i_h}{D C^i_i}.$$  

(3)

AN ALTERNATIVE MODEL FOR THE EPQ WITH PARTIAL BACKORDERING AND A LIFO POLICY
We develop here an alternative to Mak’s [1] model for the EPQ with partial backordering and a LIFO policy on filling backorders. We will use the same modeling approach used by Pentico and Drake [4,5] for the EOQ with partial backordering and the EPQ with partial backordering and a FIFO policy on filling backorders. We will show how the use of the LIFO rather than the FIFO policy changes the equations and the condition $\beta$ must meet for those equations to give an optimal solution.

The Profit and Cost Functions Based on $T$ and $F$
The first step is to develop equations for the lengths of four parts of the inventory cycle. Starting with the time that the existing inventory is exhausted, they are: 1) $t_1$ is the time until the maximum backorder level is reached and production starts, 2) $t_2$ is the time from the start of production until the backorder is eliminated and inventory starts to accumulate, 3) $t_3$ is the time until production stops and the maximum inventory level is reached, 4) $t_4$ is the time until the inventory is exhausted and a new cycle begins. Under LIFO the values for $t_1$, $t_2$, $t_3$, and $t_4$ are:

$$t_1 = \frac{(1-F)T(P-D)}{P-D(1-\beta)}, t_2 = \frac{\beta(1-F)TD}{P-D(1-\beta)}, t_3 = \frac{FTD}{P}, t_4 = FT(1 - \frac{D}{P})$$

(4)

The average profit per year to be maximized is the revenue from filling demands, either from stock or as backorders, minus the cost of placing orders, the cost of the units used or sold, the cost of carrying inventory, the cost of the backorders, and the cost of lost sales. This is maximized by the pair $(T,F)$ that minimizes the average cost per period:

$$\Gamma(T,F) = \frac{C_o}{T} + C_h \bar{I} + C_b \bar{B} + C_i D(1 - \beta)(1 - F).$$

(5)

Since $I = FTD(1 - D/P)$ and $B = \beta D t_1 = \beta D(1 - F)T(P - D)/(P - D(1 - \beta))$, the average inventory and average backorder level are:
\[ I = \frac{D T F^2}{2} (1 - \frac{D}{P}) \text{ and } B = \frac{\beta(1-F)^2 TD(P-D)}{2(P-D(1-\beta))}. \]  

(6)

Substituting the expressions for \( I \) and \( B \) into (5) gives:

\[ \Gamma(T,F) = \frac{C_o}{T} + \frac{C_h D T F^2}{2} (1 - \frac{D}{P}) + \frac{\beta C_h D T (1-F)^2 (P-D)}{2(P-D(1-\beta))} + \frac{C_i D (1-F)(1-\beta)(P-D)}{P-D(1-\beta)}. \]  

(7)

**Determining the optimal values for T and F**

Taking the partial derivative of \( \Gamma(T,F) \) in (7) with respect to \( T \) and setting it equal to 0 gives:

\[ T = \frac{2 C_o}{D \left[ D^2 C_h (1-\frac{D}{P}) (1 - \frac{D}{P}) + (1-F)^2 \beta C_h (1 - \frac{D}{P}) \right]} \]  

(8)

Taking the partial derivative of \( \Gamma(T,F) \) with respect to \( F \) and setting it equal to 0 gives:

\[ F = \frac{(1-\beta)C_i + \beta C_b T}{T(C_h (1-\frac{D(1-\beta)}{P}) + \beta C_b)} \]  

(9)

We define \( C_o^* = C_o(1-D(1-\beta)/P) \) and \( C_h^* = C_h(1-D(1-\beta)/P) \). With these replacements for \( C_o \) and \( C_h \), substituting the expression for \( F \) in (9) into equation (8), we get, after some algebra:

\[ T^* = \sqrt{\frac{2 C_o^*}{D C_h^*(1-D/P)} \left[ \frac{C_h^* + \beta C_b}{\beta C_b} \right] - \frac{[(1-\beta)C_i]^2}{\beta C_h^* C_b}} \]  

(10)

\[ F(T^*) = \frac{(1-\beta)C_i + \beta C_b T}{T^*(C_h^* + \beta C_b)} \]  

(11)

**Determining the minimum value of \( \beta \) for optimality**

Recognizing that \( T^* \) for the partial backordering model must be at least as large as \( T^* \) for the basic EPQ \( \left( \sqrt{2 C_o^*/(D C_h^*(1-D/P))} \right) \) for backordering to be optimal gives the bound:

\[ \frac{2 C_o^*}{D C_h^*(1-D/P)} \left[ \frac{C_h^* + \beta C_b}{\beta C_b} \right] - \frac{[(1-\beta)C_i]^2}{\beta C_h^* C_b} \geq \frac{2 C_o}{D C_h(1-D/P)} . \]

After some algebra, this leads us to the following conclusion: For the equations for \( T^* \) and \( F^* \) to give an optimal solution, we must have:

\[ \sqrt{\frac{2 C_o^*}{D C_h^*(1-D/P)} - \frac{(1-\beta)C_i}{C_h^*}} \text{ or } \beta > 1 - \sqrt{\frac{2 C_o^* C_h^*}{D C_h^*(1-D/P)}} . \]  

(12)

This is the same criterion derived by Mak [1]. We note that it has the same basic form as the condition derived by Park [3] and Pentico and Drake [4] for determining the minimum value of \( \beta \) for which backordering is optimal for the partial backordering EOQ.

**Procedure for finding the optimal inventory policy**

The procedure for determining the optimal values for \( T, F, Q, I, S, \) and \( B \) is, then:

1. Use (12) to determine whether \( \beta \) meets the criterion for the optimality of partial backordering.
2. a. If “No”, determine \( T^* \) from the basic EPQ model (\( T^* = \sqrt{\frac{2C_0}{C_hD(1-D/P)}} \)) and determine the optimal cost of allowing no stockouts (\( \Gamma^* = \sqrt{2C_0C_hD(1-D/P)} \)). Compare this with the cost of losing all demand, \( C_lD \), to determine whether it is optimal to allow no stockouts or all stockouts.

b. If “Yes”, use (10) to determine the value of \( T^* \) and substitute it into (11) to determine the value of \( F^* \).

3. Determine the values of the other variables as follows:

Total demand during a cycle = \( DT^* \)

Maximum inventory = \( I^* = F^*DT^*(1 - D/P) \)

Maximum stockout = \( S^* = Dt_1^* = (1 - F^*)T^*D(P - D)/(P - D(1 - \beta)) \)

Maximum backorder = \( B^* = \beta S^* \)

Order quantity = \( Q^* = DT^* - (1 - \beta)S^* \)

**CONCLUSION**

By using \( T \), the time between orders, and \( F \), the fill rate, as the decision variables, we have developed a model for the EPQ with partial backordering and a LIFO policy on filling backorders with equations that are more like those for the basic EPQ model and its full-backordering extension and are easier to solve than the equations developed by Mak [1].

**REFERENCES**


