ABSTRACT

This paper considers the basic Economic Order Quantity model that allows stockouts and backordering. It assumes that the number of defectives produced by the manufacturing process is random rather than constant. Specifically, we assume that each lot contains a random proportion of defective units. Based on this scenario, we adjust the EOQ with planned shortages model for the quality factor. In addition to the general relationships obtained, closed form relationships are also given for the two special cases where the proportion of defectives follow uniform and exponential distributions.

Keywords: Inventory Theory, Quality

INTRODUCTION

Traditionally, upon solution, independent demand inventory models result in the determination of a closed form for the economic order quantity (EOQ). Generally, this is obtained from the assumption that holding costs and setup costs are constant and equal at the optimum. Another assumption of some interest is the instantaneous replenishment of items of perfect quality. The reality of stockless production (Hall [5]) motivated researchers to give careful consideration to the practicality of these assumptions when developing new inventory models. The ultimate goal of stockless production, from an inventory standpoint, is to keep order quantities and production lots to a practical minimum. One important element of a plan aimed at achieving this goal is continuous quality improvement; since the better process quality, the lower the buffer inventory required to meet a specified service level.

Recognizing the practical importance of quality in operations, researchers have developed a number of inventory models that investigate the relationships among order quantity and quality. Rosenblatt and Lee [10] investigated the effect of process quality on lot size in the classical economic manufacturing quantity (EMQ) model. Porteus [9] introduced a modified EMQ model that indicates a significant relationship between quality and lot size. In both of these works, the optimal lot size is shown to be smaller than that of the EMQ model. In both the above cases, demand is assumed to be deterministic. Moinzadeh and Lee [6] investigated the effect of defective items on the order quantity and reorder point of a continuous-review inventory model with Poisson demand and constant lead time. Paknejad
et al [8] extend this work to consider stochastic demand and constant lead time in the continuous review (s,Q) model. Paknejad et al [7] develop a quality-adjusted EOQ model for the case where both backorders and stockouts are allowed.

Cheng [3] develops a model that integrates quality considerations with the EPQ. The author assumes that unit production cost increases with increases in process capability and quality assurance expenses. Classical optimization results in closed forms for the optimal lot size and optimal expected fraction acceptable. The optimal lot size is intuitively appealing since it indicates an inverse relationship between lot size and process capability. It should be noted that a good survey of the early literature on integrating lot size and quality control policies is given in Goyal et al [4].

In this early work the authors assume that the manufacturer operates a process that is in statistical control. That is, the process generates a known, constant proportion of defectives, p. Such an assumption induces a situation where the proportion of defective items follows a binomial distribution, and process quality, therefore, may be monitored by a proportion control chart. This assumption is also made in more recent work by Affisco et al [1] for the case of the EOQ and Affisco et al [2] for the case of the joint economic lot size model.

In the face of this concentration of research on the case where the manufacturing process is assumed to be stable, we are obligated to ask a different question. That is, what is the relationship between order quantity and quality for processes that have not yet achieved the state of statistical control? Many such processes exist in a variety of manufacturing settings, especially those that are in the initial stages of implementing a quality program. Knowledge gained from the analysis of this relationship should prove valuable in assisting managers in deciding on the proper design, implementation, and monitoring of programs aimed at improving process quality. This paper begins the investigation of inventory models for this scenario. Specifically, in the next section we consider the case of the EOQ with backorders where the number of defectives produced by the manufacturing process is random rather than constant.

**MODEL AND ASSUMPTIONS**

Consider the basic Economic Order Quantity model that allows stockouts and backordering with the following total annual cost function

\[
TC(S, Q) = \frac{D}{Q} K + \frac{(Q - S)^2}{2Q} C_h + \frac{S^2}{2Q} C_b
\]

(1)

where

- D = Annual Demand in units,
- Q = Lot size per order,
- S = Number of units backordered,
- K = Setup cost per setup,
- \(c_h\) = Holding cost per unit per year,
- \(c_b\) = Backordering cost per unit per year.
The results of classical optimization yields the following well-known expressions for the optimal values for the lot size, \( Q^* \), units backordered, \( S^* \), and the annual cost, \( C^*(S,Q) \)

\[
Q^* = \sqrt{\frac{2DK}{c_h + c_b} \left( \frac{c_h + c_h}{c_h} \right)} \tag{2}
\]

\[
S^* = Q^* \left( \frac{c_h}{c_h + c_b} \right) \tag{3}
\]

and

\[
C^*(S,Q) = \sqrt{\frac{2DK}{1 + \frac{1}{c_h + c_b}}} \tag{4}
\]

Implicit in these derivations is that all units produced by the vendor, in response to the purchaser’s order, are of acceptable quality. Now, assume that this is not the case. Specifically, assume that each lot contains a random proportion of defective units. Upon arrival, the purchaser inspects the entire lot. We further assume that the purchaser’s inspection process is perfect, and that all rejected items are returned to the vendor at no cost to the purchaser. In addition, we assume that the inspection cost is paid by the vendor. Of course, it is likely that the vendor will recover some of these costs from the purchaser either directly or indirectly. Based on this scenario, we now adjust the EOQ with planned shortages model for the quality factor.

Let
\[
\theta = \text{Proportion of defective items in a lot, a continuous random variable,}
\]
\[
f(\theta) = \text{Probability density function of } \theta,
\]
\[
y = (1-\theta)Q = \text{Number of non-defective items in a lot,}
\]
\[
c(y) = \text{Cost per cycle given that there are } y \text{ non-defective items in the lot,}
\]
\[
T = y/D = \text{Cycle time (time between two successive placement of orders),}
\]
\[
E(y) = \text{First moment of } y,
\]
\[
E(y^2) = \text{Second moment of } y \text{ given a lot of size } Q \text{ is ordered,}
\]
\[
E(c) = \text{Expected cycle cost per year,}
\]
\[
E(T) = \text{Expected value of } T \text{ given a lot of size } Q \text{ is ordered,}
\]
\[
C(S,Q) = \text{Expected total cost per year.}
\]

The total cost per cycle is

\[
c(y) = K + \frac{(y-S)^2}{2D}c_h + \frac{S^2}{2D}c_b = K + \left[ \frac{(1-\theta)Q-S}{2D} \right]^2c_h + \frac{S^2}{2D}c_b \tag{5}
\]

The average cycle time and cycle cost are
\[ E(T) = \frac{E(y)}{D} = \frac{E[(1-\theta)Q]}{D} = \frac{Q[1-E(\theta)]}{D} \]  
(6)

and

\[ E(c) = K + \frac{c_h Q}{2D} \left[ Q\left[1-2E(\theta)+E(\theta^2)\right]-2S\left[1-E(\theta)\right]\right] + \frac{c_h + c_b}{2D} S^2 \]  
(7)

The expected total annual cost is

\[ c(S, Q) = \frac{DK}{Q[1-E(\theta)]} + \left[ \frac{Q\left[1-2E(\theta)+E(\theta^2)\right]-2S\left[1-E(\theta)\right]}{2\left[1-E(\theta)\right]} - S \right] + \frac{(c_h + c_b) S^2}{2Q\left[1-E(\theta)\right]} \]  
(8)

The optimal values for the order quantity, \( Q^*_{adj} \), units backordered, \( S^*_{adj} \), and expected total annual cost, \( c^*_{adj}(S, Q) \), are found by using calculus as follows

\[
Q^*_{adj} = \frac{1}{1-E(\theta)} \sqrt{\frac{2DK}{c_h \left[1-2E(\theta)+E(\theta^2)\right] - \frac{c_h}{c_h + c_b}}}
\]  
(9)

\[
S^*_{adj} = \left[1-E(\theta)\right] \left(\frac{c_h}{c_h + c_b}\right) Q^*_{adj}
\]  
(10)

and

\[
c^*_{adj} = \frac{E(\theta)}{2} c_h + \sqrt{2DKc_h \left[1-2E(\theta)+E(\theta^2)\right] - \frac{c_h}{c_h + c_b}}
\]  
(11)

Note that in equation (9), (10), and (11), when \( \theta \) is constant, then the number of non-defective units in a lot is binomial and the general quality-adjusted EOQ with planned shortage model with random proportion of defective units of this paper simply reduces to the model of Paknejad, Nasri, Affisco [7]. Furthermore, if \( \theta = 0 \), then the quality is perfect and equations (9), (10), and (11) simply reduce to the corresponding results of the basic EOQ with planned shortage model expressed by equations (2), (3), and (4).

Let us now consider a few additional special cases of \( \theta \) as a continuous random variable. When \( \theta \) is uniformly distributed between 0 and \( t \), then

\[
f(\theta) = \begin{cases} \frac{1}{t} & \text{for } 0 \leq \theta \leq t \\ 0 & \text{elsewhere} \end{cases}
\]  
(12)

In this case

\[
Q^*_{adj,u} = \frac{2}{2-t} \sqrt{\frac{2DK}{c_h \left[4(3-3t+t^2)\right] - \frac{c_h}{3(4-4t+t^2)}}}
\]  
(13)
\[ S_{adj,\mu}^* = \frac{2}{2-t} \left( \frac{c_h}{c_h + c_b} \right) Q_{adj,\mu}^* , \] (14)

and

\[ C_{adj,\mu}^* = \frac{t}{4} c_h + \sqrt{2DKc_h \left[ \frac{4(3t+t^2)}{3(4t^2 + t)} - \frac{c_h}{c_h + c_b} \right]} . \] (15)

When \( \theta \) has an exponential distribution \([0,1]\), then

\[ f(\theta) = \frac{e}{e-1} e^{-\theta} \quad \text{for} \quad 0 \leq \theta \leq 1 \] (16)

\[ E(\theta) = \frac{e}{e-1} \int_0^1 \theta e^{-\theta} d\theta = \frac{e}{e-1} \frac{e-2}{e-1} \] (17)

\[ E(\theta^2) = \frac{e}{e-1} \int_0^1 \theta e^{-\theta} d\theta = \frac{2e^2 - 5}{e-1} \] (18)

\[ Q_{adj,e}^* = (e-1) \sqrt{\frac{2DK}{c_h \left[ (e-1)(e-2) - \frac{c_h}{c_h + c_b} \right]}} , \] (19)

\[ S_{adj,e}^* = \frac{1}{e-1} \left( \frac{c_h}{c_h + c_b} \right) Q_{adj,e}^* , \] (20)

\[ C_{adj,e}^* = \frac{1}{2} \left( \frac{e-2}{e-1} \right) c_h + \sqrt{2DKc_h \left[ (e-1)(e-2) - \frac{c_h}{c_h + c_b} \right]} . \] (21)

**CONCLUSION**

This paper opens a new line of inquiry into order quantity models under the situation of imperfect quality. Closed forms are derived for the case where the proportion of defectives is randomly distributed. Hence, the manufacturing process is unstable with no known process capability. In addition to the general relationships, closed forms are also given for the two special cases where the proportion of defectives follows a uniform distribution and an exponential distribution.

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REFERENCES


